A stability-like theorem for cohomology of Pure Braid Groups of the series A, B and D

Simona Settepanella

February 1, 2008

ABSTRACT. Consider the ring $R := \mathbb{Q}[\tau, \tau^{-1}]$ of Laurent polynomials in the variable τ. The Artin's Pure Braid Groups (or Generalized Pure Braid Groups) act over R, where the action of every standard generator is the multiplication by τ. In this paper we consider the cohomology of such groups with coefficients in the module R (it is well known that such cohomology is strictly related to the untwisted integral cohomology of the Milnor fibration naturally associated to the reflection arrangement). We give a sort of *stability* theorem for the cohomologies of the infinite series A, B and D, finding that these cohomologies stabilize, with respect to the natural inclusion, at some number of copies of the trivial R-module $\mathbb{Q}$. We also give a formula which computes this number of copies.

1 Introduction

Let $(\mathbf{W}, S)$ be a finite Coxeter system realized as a reflection group in $\mathbb{R}^n$, $\mathcal{A}(\mathbf{W})$ the arrangement in $\mathbb{C}^n$ obtained by complexifying the reflection hyperplanes of $\mathbf{W}$.Let

$$\mathbf{Y}(\mathbf{W}) = \mathbf{Y}(\mathcal{A}(\mathbf{W})) = \mathbb{C}^n \setminus \cup_{H \in \mathcal{A}(\mathbf{W})} H.$$

be the complement to the arrangement, then $\mathbf{W}$ acts freely on $\mathbf{Y}(\mathbf{W})$ and the fundamental group G_W of the orbit space $\mathbf{Y}(\mathbf{W})/\mathbf{W}$ is the so called *Artin group* associated to $\mathbf{W}$ (see [**?**]). Likewise the fundamental group P_W of $\mathbf{Y}(\mathbf{W})$ is the *Pure Artin group* or the pure braid group of the series $\mathbf{W}$. It is well known ([**?**]) that these spaces $\mathbf{Y}(\mathbf{W})$ ($\mathbf{Y}(\mathbf{W})/\mathbf{W}$) are of type $K(\pi, 1)$, so there cohomologies equal that of P_W (G_W).

The integer cohomology of $\mathbf{Y}(\mathbf{W})$ is well known (see [**?**],[**?**], [**?**],[**?**]) and so is the integer cohomology of the Artin groups associated to finite Coxeter groups (see [**?**],[**?**],[**?**]).

Let $R = \mathbb{Q}[\tau, \tau^{-1}]$ be the ring of rational Laurent polynomials. The R can be given a structure of module over the Artin group G_W, where standard generators of G_W act as τ-multiplication.

In [**?**] and [**?**] the authors compute the cohomology of all Artin groups associated to finite Coxeter groups with coefficients in the previous module.

In a similar way we define a P_W-module R_τ, where standard generators of P_W act over the ring R as τ-multiplication.

Equivalently, one defines an abelian local system (also called R_τ) over $\mathbf{Y}(\mathbf{W})$ with fiber R and local monodromy around each hyperplane given by τ-multiplication (for local systems on $\mathbf{Y}(\mathbf{W})$ see [**?**],[**?**]).

In this paper we are going to consider the cohomology of $\mathbf{Y}(\mathbf{W})$ with local coefficients R_τ, for the finite Coxeter groups of the series A, B and D (see [**?**]) (that is equivalent to the cohomology of P_W with coefficients in R_τ).

Our aim is to give a sort of *"stability"* theorem for these cohomologies (for stability in the case of Artin groups see [**?**]) .

Denote by φ_i the i-th cyclotomic polynomial and let be

$$\{\varphi_i\} := \mathbb{Q}[\tau, \tau^{-1}]/(\varphi_i) = \mathbb{Q}[\tau]/(\varphi_i)$$

thought as R-module.By its definition $\{\varphi_1\} = 1 - \tau$ so that $\{\varphi_1\} = \mathbb{Q}$.

Notice that by identification $\mathbb{Q}[\tau, \tau^{-1}] \cong \mathbb{Q}[\mathbb{Z}]$, the sums of copies of $\{\varphi_1\}$ are the unique trivial $\mathbb{Z}$-modules. We obtain

Theorem 1.1. *Let* $\mathbf{W}$ *be a Coxeter group of type* A_n*, then for* $n \geq 3k-2$ *the cohomology group* $H^k(\mathbf{Y}(A_n), R_\tau)$ *is a trivial* $\mathbb{Z}$*-module.*
Analog statement holds for $\mathbf{W}$ *of type* B_n *in the rang* $n \geq 2k-1$ *and for* $\mathbf{W}$ *of type* D_n *in the rang* $n \geq 3k-1$*.*

The proof of this theorem is obtained extending the methods developed in [?] and using some known results about the global Milnor fibre $F(\mathbf{W})$ of the complement $\mathbf{Y}(\mathbf{W})$.

We recall briefly that if $H \in \mathcal{A} = \mathcal{A}(\mathbf{W})$ and $\alpha_H \in \mathbb{C}[x_1, \ldots, x_n]$ is a linear form s.t. $H = ker(\alpha_H)$, then the global Milnor fibre $F(\mathbf{W})$ is a complex manifold of dimension $n-1$ given by $F(\mathbf{W}) = Q^{-1}(1)$ where $Q = Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$ is the *defining polynomial* for $\mathcal{A}$.

It is well known (see also [?]) that, over R, there is a decomposition

$$H^*(F(\mathbf{W}), \mathbb{Q}) \simeq \bigoplus_{i|\sharp(\mathcal{A}(\mathbf{W}))} (R/(\varphi_i))^{\alpha_i} = \bigoplus_{i|\sharp(\mathcal{A}(\mathbf{W}))} \{\varphi_i\}^{\alpha_i}.$$

the action on the left being that induced by monodromy.

Since $F(\mathbf{W})$ is homotopy-equivalent to an infinite cyclic cover of $\mathbf{Y}(\mathbf{W})$, there is an isomorphism of R-modules

$$H^*(F(\mathbf{W}), \mathbb{Q}) \simeq H^*(\mathbf{Y}(\mathbf{W}), R_\tau)$$

and then

$$H^*(\mathbf{Y}(\mathbf{W}), R_\tau) \simeq \bigoplus_{i|\sharp(\mathcal{A}(\mathbf{W}))} \{\varphi_i\}^{\alpha_i}. \tag{1}$$

The other tool we use is a suitable filtration by subcomplexes of the algebraic Salvetti's CW-complex ($C(\mathbf{W})$,δ) coming from [?] (see also [?], [?]), which we recall in the next paragraph.

Finally we use the universal coefficients theorem to compute the dimensions of the above cohomologies as vector spaces over the rationals.

Theorem 1.2. *In the range specified in theorem 1.1 one has:*

$$\mathtt{rk} H^{k+1}(Y(\mathbf{W}), R_\tau) = \sum_{i=0}^{k} (-1)^{(k-i)} \mathtt{rk} H^i(Y(\mathbf{W}), \mathbb{Z}).$$

So one reduces to compute the dimensions of the Orlik-Solomon algebras of $\mathcal{A}(\mathbf{A_n})$, $\mathcal{A}(\mathbf{B_n})$ and $\mathcal{A}(\mathbf{D_n})$ (see [?]).

2 Salvetti's Complex

Let $\mathbf{W}$ be a finite group generated by reflections in the affine space $\mathbb{A}^n(\mathbb{R})$. Let $\overline{\mathcal{A}}(\mathbf{W}) = \{H_j\}_{j\in J}$ be the arrangement in $\mathbb{A}^n$ defined by the reflection hyperplanes of $\mathbf{W}$. We need to recall briefly some notations and results from [?] for the particular case of Coxeter arrangements. $\overline{\mathcal{A}}(\mathbf{W})$ induces a stratification $\mathcal{S} = \mathcal{S}(\mathbf{W})$ of $\mathbb{A}^n$ into facets (see [?]). The set $\mathcal{S}$ is partially ordered by $F > F'$ iff $F' \subset \mathrm{cl}(F)$. We shall indicate by $\mathbf{Q} = \mathbf{Q}(\mathbf{W})$ the cellular complex which is *dual* to $\mathcal{S}$. In a standard way, this can be realized inside $\mathbb{A}^n$ by baricentrical subdivision of the facets: inside each codimension j facet F^j of $\mathcal{S}$ choose one point $v(F^j)$ and consider the simplexes

$$s(F^{i_0},\cdots,F^{i_j}) = \{\sum_{k=0}^{j} \lambda_k v(F^{i_k}) : \sum_{k=0}^{j} \lambda_k = 1, \lambda_k \in [0,1]\}$$

where $F^{i_{k+1}} < F^{i_k}, k = 0,\cdots,j-1$. The dimension j cell $e^j(\overline{F}^j)$ which is dual to $\overline{F}^j$ is obtained by taking the union

$$\cup s(F^0,\cdots,F^{j-1},\overline{F}^j)$$

over all chains $\overline{F}^j < F^{j_1} < \cdots < F^0$. So $\mathbf{Q} = \cup e^j(F^j)$, the union being over all facets of $\mathcal{S}$.

One can think of the $1-$*skeleton* $\mathbf{Q}_1$ as a graph (with vertex-set the $0-$*skeleton* $\mathbf{Q}_0$) and can define the combinatorial distance between two vertices v, v' as the minimum number of edges in an edge-path connecting v and v'.

For each cell e^j of $\mathbf{Q}$ one indicates by $V(e^j) = \mathbf{Q}_0 \cap e^j$ the 0-*skeleton* of e^j. One has

Proposition 2.1. *Given a vertex $v \in \mathbf{Q}_0$ and a cell $e^i \in \mathbf{Q}$, there is a unique vertex $\underline{w}(v, e^i) \in V(e^i)$ with the lowest combinatorial distance from v, i.e.:*

$$d(v, \underline{w}(v, e^i)) < d(v, v') \text{ if } v' \in V(e^i) \setminus \{\underline{w}(v, e^i)\}.$$

If $e^j \subset e^i$ then $\underline{w}(v, e^j) = \underline{w}(\underline{w}(v, e^i), e^j)$.

Let now $\mathcal{A}(\mathbf{W})$ denote the *complexification* of $\overline{\mathcal{A}}(\mathbf{W})$, and $\mathbf{Y}(\mathbf{W}) = \mathbb{C}^n \setminus \cup_{j\in J} H_{j,\mathbb{C}}$ the complement of the complexified arrangement. Then $\mathbf{Y}(\mathbf{W})$ is homotopy equivalent to the complex $\mathbf{X}(\mathbf{W})$ which is constructed as follows (see [**?**]).

Take a cell $e^j = e^j(F^j) = \cup s(F^0, \cdots, F^{j-1}, F^j)$ of $\mathbf{Q}$ as defined above and let $v \in V(e^j)$. Embed each simplex $s(F^0, \cdots, F^j)$ into $\mathbb{C}^n$ by the formula

$$\begin{aligned}\phi_{v,e_j}\Big(\sum_{k=0}^{j} \lambda_k v(F^k)\Big) = \\ \sum_{k=0}^{j} \lambda_k v(F^k) + i \sum_{k=0}^{j} \lambda_k (\underline{w}(v, e^k) - v(F^k)).\end{aligned} \tag{2}$$

It is shown in [**?**] (see also [**?**]):

(i) the preceding formula defines an embedding of e^j into $\mathbf{Y}(\mathbf{W})$;

(ii) if $E^j = E^j(v, e^j)$ is its image, then varying e^j and v one obtains a cellular complex

$$\mathbf{X}(\mathbf{W}) = \cup E^j$$

which is homotopy equivalent to $\mathbf{Y}(\mathbf{W})$.

The previous result allows us to make cohomological computations over $\mathbf{Y}(\mathbf{W})$ by using the complex $\mathbf{X}(\mathbf{W})$.

In [**?**] (see also [**?**]) the authors give a new combinatorial description of the stratification $\mathcal{S}$ where the action of $\mathbf{W}$ is more explicit. They prove that if S is the set of reflections with respect to the walls of the fixed base chamber C_0, then a cell in $\mathbf{X}(\mathbf{W})$ is of the form $E = E(w, \Gamma)$ with $\Gamma \subset S$ and $w \in \mathbf{W}$. The action of $\mathbf{W}$ is written as

$$\sigma.E(w, \Gamma) = E(\sigma w, \Gamma), \tag{3}$$

where the factor $\sigma.w$ is just multiplication in $\mathbf{W}$.

We prefer at the moment to deal with chain complexes and boundary operator coming from $\mathbf{X}(\mathbf{W})$ instead of cochain and coboundary. Then we will deduce cohomological results by standard methods.

We define a rank-1 local system on $\mathbf{Y}(\mathbf{W})$ with coefficients in an unitary ring A by assigning an unit $\tau_j = \tau(H_j)$ (thought as a multiplicative operator) to each hyperplane $H_j \in \mathcal{A}$. Call $\overline{\tau}$ the collection of τ_j and $\mathcal{L}_{\overline{\tau}}$ the corresponding local system. Let $C(\mathbf{W}, \mathcal{L}_{\overline{\tau}})$ be the free graduated A-module with basis all $E(w, \Gamma)$.

We use the natural identification between the elements of the group and the vertices of $\mathbf{Q}_0$, given by $w \leftrightarrow w.v_0$. Here $v_o \in \mathbf{Q}_0$ is contained in the fixed base chamber C_0.

Then $u(w, w')$ will denote the *"minimal positive path"* joining the corresponding vertices v and v' in the 1-skeleton $\mathbf{X}(\mathbf{W})_1$ of $\mathbf{X}(\mathbf{W})$ (see [**?**]).

The local system $\mathcal{L}_{\overline{\tau}}$ defines for each edge-path c in $\mathbf{X}(\mathbf{W})_1$, $c : w \to w'$ an isomorphism $c_* : A \to A$ such that for all $d : w \to w'$ homotopic to c, $c_* = d_*$ and for all $f : w'' \to w$, $(cf)_* = c_* f_*$.

Then the set $\{s_0(w).E(w, \Gamma)\}_{|\Gamma|=k}$, where $s_0(w) := u(1, w)_*(1)$, is a linear basis of $C_k(\mathbf{W}, \mathcal{L}_{\overline{\tau}})$.

Let now $T = \{wsw^{-1} | s \in S, w \in \mathbf{W}\}$, the set of reflections in $\mathbf{W}$ and

$$\overline{\mathbf{W}} = \{\mathbf{s}(w) = (s_{i_1}, \cdots, s_{i_q}) | w = s_{i_1} \cdots s_{i_q} \in \mathbf{W}\},$$

then for each $\mathbf{s}(w) \in \overline{\mathbf{W}}$ and $t \in T$, we set

i) $\Psi(\mathbf{s}(w)) = (t_{i_1}, \cdots, t_{i_q})$ with $t_{i_j} = (s_{i_1} \cdots s_{i_{j-1}}) s_{i_j} (s_{i_1} \cdots s_{i_{j-1}})^{-1} \in T$

ii) $\overline{\Psi(\mathbf{s}(w))} = \{t_{i_1}, \cdots, t_{i_q}\}$

iii)$\eta(w, t) = (-1)^{n(\mathbf{s}(w), t)}$ with $n(\mathbf{s}(w), t) = \sharp\{j | 1 \le j \le q \text{ and } t_{i_j} = t\}$.

Moreover if $t \in T$ is the reflection relative to the hyperplane H, then we set $\tau(t) = \tau(H)$.

We define

$$\partial_k(s_0(w).E(w,\Gamma)) = \sum_{\sigma\in\Gamma}\sum_{\beta\in\mathbf{W}_\Gamma^{\Gamma\setminus\{\sigma\}}} (-1)^{l(\beta)+\mu(\Gamma,\sigma)}\tau(w,\beta)s_0(w\beta).E(w\beta,\Gamma\setminus\{\sigma\}). \quad (4)$$

where $\tau(w,\beta) = \prod_{\substack{t\in\Psi(\mathbf{s}(w))\\ \eta(w,t)=1}} \tau(t)$, and $\mu(\Gamma,\sigma) = \sharp\{i\in\Gamma | i\leq\sigma\}$.

We have the following (see [**?**], [**?**])

Theorem 2.1. *$H_*(C(\mathbf{W}),\mathcal{L}_{\overline{\tau}}) \cong H_*(C(\mathbf{W},\mathcal{L}_{\overline{\tau}}),\partial)$.*

We have a similar result for the cohomology.

3 A filtration for the complex $(C(\mathbf{W}),\partial)$

Let $(\mathbf{W},S)$ be a finite Coxeter system with $S=\{s_1,\cdots,s_n\}$. We are interested in the cohomology of $C(\mathbf{W})$ (equivalently $\mathbf{Y}(\mathbf{W})$) with coefficients in R_τ (see introduction).

In this case the boundary operator defined in (4) becomes

$$\partial(E(w,\Gamma)) = \sum_{\sigma\in\Gamma}\sum_{\beta\in\mathbf{W}_\Gamma^{\Gamma\setminus\{\sigma\}}} (-1)^{l(\beta)+\mu(\Gamma,\sigma)}\tau^{\frac{l(\beta)+l(w)-l(w\beta)}{2}}E(w\beta,\Gamma\setminus\{\sigma\}) \quad (5)$$

where τ is the variable in the ring R.

From (1) and universal coefficients theorem it follows that

$$H^*(C(\mathbf{W}),R_\tau) = H_{*-1}(C(\mathbf{W}),R_\tau). \quad (6)$$

For each integer $0\leq k\leq n$ denote by $S_k=\{s_1,\cdots,s_k\}\subset S$ and $S^k = S\setminus S_k$. We define the graduated R-submodules of $C(\mathbf{W})$:

$$G_n^k(\mathbf{W}) := \sum_{\substack{w\in\mathbf{W}\\ \Gamma\subset S_k}} R.E(w,\Gamma)$$

$$F_n^k(\mathbf{W}) := \sum_{\substack{w\in\mathbf{W}\\ \Gamma\supset S^{n-k}}} R.E(w,\Gamma).$$

There is an obvious inclusion

$$i_{n,h} : G_n^{n-h}(\mathbf{W}) \longrightarrow G_n^n(\mathbf{W}) = C(\mathbf{W}). \tag{7}$$

Each $G_n^k(\mathbf{W})$ is preserved by the induced boundary map and we get a filtration by subcomplexes of $C(\mathbf{W})$:

$$C(\mathbf{W}) = G_n^n(\mathbf{W}) \supset G_n^{n-1}(\mathbf{W}) \cdots \supset G_n^1(\mathbf{W}) \supset G_n^0(\mathbf{W}).$$

The quotient module $G_n^n(\mathbf{W})/G_n^{n-1}(\mathbf{W})$ is exactly $F_n^1(\mathbf{W})$ which becomes an algebraic complex with the induced boundary map.

We give iteratively to $F_n^k(\mathbf{W})$, $k \geq 2$, a structure of complex by identifying it with the cokernel of the map:

$$\begin{aligned} &i_n[k] : G_n^{n-(k+1)}(\mathbf{W})[k] \longrightarrow F_n^k(\mathbf{W}), \\ &i(E(w,\Gamma))) = E(w, \Gamma \cup S^{n-k}). \end{aligned}$$

Here $M[k]$ denotes, as usual, k-augmentation of a complex M; so $i_n[k]$ is degree preserving.

By construction $i_n[k]$ gives rise to the exact sequence of complexes

$$0 \longrightarrow G_n^{n-(k+1)}(\mathbf{W})[k] \longrightarrow F_n^k(\mathbf{W}) \longrightarrow F_n^{k+1}(\mathbf{W}) \longrightarrow 0. \tag{8}$$

Let $\Gamma \subset S$ and let $\mathbf{W}_\Gamma$ be the *parabolic subgroup* of $\mathbf{W}$ generated by Γ. Recall from [**?**] the following

Proposition 3.1. *Let $(\mathbf{W}, S)$ be a Coxeter system. Let $\Gamma \subset S$. The following statements hold.*

(i) $(\mathbf{W}_\Gamma, \Gamma)$ is a Coxeter system.

(ii) Viewing $\mathbf{W}_\Gamma$ as a Coxeter group with length function ℓ_Γ, $\ell_S = \ell_\Gamma$ on $\mathbf{W}_\Gamma$.

(iii) Define $\mathbf{W}^\Gamma \overset{\text{def}}{=} \{w \in \mathbf{W} | \ell(ws) > \ell(w) \text{ for all } s \in \Gamma\}$. Given $w \in \mathbf{W}$, there is a unique $u \in \mathbf{W}^\Gamma$ and a unique $v \in \mathbf{W}_\Gamma$ such that $w = uv$. Their lengths satisfy $\ell(w) = \ell(u) + \ell(v)$. Moreover, u is the unique element of shortest length in the coset $w\mathbf{W}_\Gamma$.

For all $w \in \mathbf{W}$ we set $w = w^\Gamma w_\Gamma$ with $w^\Gamma \in \mathbf{W}^\Gamma$ and $w_\Gamma \in \mathbf{W}_\Gamma$. Then if $\beta \in \mathbf{W}_\Gamma$ one has $l(w\beta) = l(w^\Gamma) + l(w_\Gamma\beta)$.

From (5) it follows:

$$\partial(E(w,\Gamma)) = w^\Gamma.\partial(E(w_\Gamma,\Gamma)) \tag{9}$$

where the action (3) is extended to $C(\mathbf{W})$ by linearity.

As a consequence we have a direct sum decomposition into isomorphic factors:

$$H_q(G_n^k, R_\tau) \simeq \bigoplus_{j=1}^{|\mathbf{W}^{S_k}|} H_q(C(\mathbf{W}_{S_k}), R_\tau). \tag{10}$$

4 Preparation for the Main Theorem

Let $m_k :=| \mathbf{W}^{S_k} |$ and $\mathbf{W}_k := \mathbf{W}_{S_k}$; the exact sequences (8) with relations (10) give rise to the corresponding long exact sequences in homology

$$\begin{multline} \cdots \longrightarrow H_{q+1}(F_n^{k+1}(\mathbf{W}), R_\tau) \longrightarrow \bigoplus_{j=1}^{m_{n-k-1}} H_{q-k}(C(\mathbf{W}_{S_{n-k-1}}), R_\tau) \longrightarrow \\ \longrightarrow H_q(F_n^k(\mathbf{W}), R_\tau) \longrightarrow H_q(F_n^{k+1}(\mathbf{W}), R_\tau) \longrightarrow \cdots . \end{multline} \tag{11}$$

We have the following

Lemma 4.1. *If $H_{q-h}(C(\mathbf{W}_{n-h-1}), R_\tau)$ are trivial $\mathbb{Z}$-modules for all h such that $k \leq h \leq q$, then $H_q(F_n^k(\mathbf{W}), R_\tau)$ is also trivial.*

Proof: From (8) and (10) one has the exact sequences of complexes

$$0 \longrightarrow \bigoplus_{j=1}^{m_{n-k-1}} C(\mathbf{W}_{n-k-1})[k] \longrightarrow F_n^k(\mathbf{W}) \longrightarrow F_n^{k+1}(\mathbf{W}) \longrightarrow 0$$
$$0 \longrightarrow \bigoplus_{j=1}^{m_{n-k-2}} C(\mathbf{W}_{n-k-2})[k+1] \longrightarrow F_n^{k+1}(\mathbf{W}) \longrightarrow F_n^{k+2}(\mathbf{W}) \longrightarrow 0 \qquad (12)$$
$$\dots$$
$$0 \longrightarrow \bigoplus_{j=1}^{m_{n-q-1}} C(\mathbf{W}_{n-q-1})[q] \longrightarrow F_n^q(\mathbf{W}) \longrightarrow F_n^{q+1}(\mathbf{W}) \longrightarrow 0$$

The last sequence gives rise to the long exact sequence in homology:

$$\cdots \longrightarrow \bigoplus_{j=1}^{m_{n-q-1}} H_0(C(\mathbf{W}_{n-q-1}), R_\tau) \longrightarrow H_q(F_n^q(\mathbf{W}), R_\tau) \longrightarrow 0. \qquad (13)$$

By hypothesis $H_0(C(\mathbf{W}_{n-q-1}), R_\tau)$ is a trivial $\mathbb{Z}$-module then $H_q(F_n^q, R_\tau)$ is also trivial.

We get the thesis going backwards in (12) and considering, in a similar way of (13), the long exact sequences induced. $\square$

Recall (see (1)) the decomposition:

$$H_*(C(\mathbf{W}), R_\tau) = \bigoplus_{r|\sharp(\mathcal{A}(\mathbf{W}))} [R/(\varphi_r)]^{\alpha_r}.$$

It follows that if $\sharp(\mathcal{A}(\mathbf{W}))$ and $\sharp(\mathcal{A}(\mathbf{W}_{n-h}))$ are coprimes, the maps $i_{n,h}$ of (7) give rise to homology maps with images sums of copies of $\{\varphi_1\}$ ($\{\varphi_1\}^0$ means that the map is identically 0).

We have that $\sharp(\mathcal{A}(\mathbf{A_n})) = n(n+1)/2$ and $\sharp(\mathcal{A}(\mathbf{B_n})) = n^2$ (see [**?**]). If we fix

$$(n,h) = (3q+1, 2) \text{ for } \mathbf{A_n}$$
$$(n,h) = (n, 1) \text{ for } \mathbf{B_n}$$

then

$$(\sharp(\mathcal{A}(\mathbf{A_{3q+1}})),\sharp(\mathcal{A}(\mathbf{A_{3q-1}}))) = 1$$
$$(\sharp(\mathcal{A}(\mathbf{B_n})),\sharp(\mathcal{A}(\mathbf{B_{n-1}}))) = 1.$$

Since $i_{n,h}$ are injective, we can complete (7) to short exact sequences of complexes which give, by the above remark:

$$0 \longrightarrow \oplus\{\varphi_1\} \longrightarrow H_q(C(\mathbf{A_{3q+1}}),R_\tau) \longrightarrow H_q(C(\mathbf{A_{3q+1}})/\oplus_{j=1}^{m_{3q-1}} C(\mathbf{A_{3q-1}}),R_\tau) \longrightarrow \bigoplus_{j=1}^{m_{3q-1}} H_{q-1}(C(\mathbf{A_{3q-1}}),R_\tau) \longrightarrow \oplus\{\varphi_1\} \longrightarrow \cdots \tag{14}$$

in case $\mathbf{A_n}$ and

$$0 \longrightarrow \oplus\{\varphi_1\} \longrightarrow H_q(C(\mathbf{B_n}),R_\tau) \longrightarrow H_q(C(\mathbf{B_n})/\oplus_{j=1}^{m_{n-1}} C(\mathbf{B_{n-1}}),R_\tau) \longrightarrow \bigoplus_{j=1}^{m_{n-1}} H_{q-1}(C(\mathbf{B_{n-1}}),R_\tau) \longrightarrow \oplus\{\varphi_1\} \longrightarrow \cdots . \tag{15}$$

in case $\mathbf{B_n}$.

In order to prove theorem 1.1, we need to study the complexes $C(\mathbf{A_{3q+1}})/\oplus_{j=1}^{m_{3q-1}} C(\mathbf{A_{3q-1}})$ and $C(\mathbf{B_n})/\oplus_{j=1}^{m_{n-1}} C(\mathbf{B_{n-1}})$.

The latter is exactly the complex $F_n^1(\mathbf{B_n})$.

The farmer is the complex with basis over R:

$$\mathcal{E}_T := \{E(w,\Gamma\cup T) \mid w \in \mathbf{A_{3q+1}} \text{ and } \Gamma \subset S_{3q-1}\}$$

for $\emptyset \subsetneq T \subset S^{3q-1}$. We remark that $\mathcal{E}_{\{s_{3q}\}}$ is the basis of a complex isomorphic to $(3q+2)$ copies of $F_{3q}^1(\mathbf{A_{3q}})$, $\mathcal{E}_{\{s_{3q+1}\}}$ generates the subcomplex given by the image of $G_{3q+1}^{3q-1}(\mathbf{A_{3q+1}})$ by the map $i_{3q+1}[1]$ and the elements of $\mathcal{E}_{\{s_{3q+1},s_{3q}\}}$ are the generators of the module $F_{3q+1}^2(\mathbf{A_{3q+1}})$.

Now we set

$$(F_n^k(\mathbf{W}))_h := \{E(w,\Gamma) \in F_n^k(\mathbf{W}) \mid\mid \Gamma \mid= h\}$$

and $\partial^k_{n,h} : (F^k_n(\mathbf{W}))_h \longrightarrow (F^k_n(\mathbf{W}))_{h-1}$ the h-th boundary map in $F^k_n(\mathbf{W})$ ($\partial_{n,h} := \partial^0_{n,h}$ is the boundary map in $C(\mathbf{W})_h$).

Then the h-th boundary matrix of $C(\mathbf{A_{3q+1}})/\oplus_{j=1}^{m_{3q-1}} C(\mathbf{A_{3q-1}})$ is of the form

$$\overline{\partial}_h = \begin{bmatrix} \oplus_{i=1}^{3q+2}\partial^1_{3q,h} & 0 & A_1 \\ 0 & \oplus_{i=1}^{\frac{(3q+1)(3q+2)}{2}}\partial_{3q-1,h-1} & A_2 \\ 0 & 0 & \partial^2_{3q+1,h} \end{bmatrix}$$

where A_1 and A_2 are the matrices of the image of the generators in $\mathcal{E}_{\{s_{3q},s_{3q+1}\}}$ restricted to $\mathcal{E}_{\{s_{3q}\}}$ and $\mathcal{E}_{\{s_{3q+1}\}}$ respectively.

Moreover all homology groups of the complexes $F^k_n(\mathbf{W})$ are torsion groups so the rank of $\partial^k_{n,h}$ equals the rank of $\ker(\partial^k_{n,h-1})$. Then it is not difficult to see that the rank of $\overline{\partial}_h$ is exactly the sum of $(3q+2)$ times the rank of $\partial^1_{3q,h}$, $\frac{(3q+1)(3q+2)}{2}$ times the rank of $\partial_{3q-1,h-1}$ and the rank of $\partial^2_{3q+1,h}$.

Remark 4.1. *It follows that in order to prove that* $H_k(C(\mathbf{A_{3q+1}})/\oplus_{j=1}^{m_{3q-1}} C(\mathbf{A_{3q-1}}), R_\tau)$ *is sum of copies of* $\{\varphi_1\}$*, i.e. a trivial* $\mathbb{Z}$*-module, it is sufficient to prove the same result for* $H_k(F^1_{3q}(\mathbf{A_{3q}}), R_\tau)$*,* $H_{k-1}(C(\mathbf{A_{3q-1}}), R_\tau)$ *and* $H_k(F^2_{3q+1}(\mathbf{A_{3q+1}}), R_\tau)$*.*

5 Proof of the Main Theorem

In this section we prove theorem 1.1. This is equivalent to prove that $H_k(C(\mathbf{A_n}), \mathbb{R}_\tau)$ is a trivial $\mathbb{Z}$-module for $n \geq 3k+1$, $H_k(C(\mathbf{B_n}), \mathbb{R}_\tau)$ is trivial for $n \geq 2k+1$ and $H_k(C(\mathbf{D_n}), \mathbb{R}_\tau)$ is trivial for $n \geq 3k+2$ (see relation (6)).

For cases $\mathbf{A_n}$ and $\mathbf{B_n}$ we use induction on the degree of homology. Case $\mathbf{D_n}$ will follow from $\mathbf{A_n}$.

By standard methods (see also [**?**]) one gets the first step of induction, which is

$$H_0(C(\mathbf{A_n}), R_\tau) \simeq H_0(C(\mathbf{B_n}), R_\tau) \simeq \{\varphi_1\} \tag{16}$$

for all $n \geq 1$.

One supposes that $H_{k-1}(C(\mathbf{A_n}), R_\tau)$ and $H_{k-1}(C(\mathbf{B_n}), R_\tau)$ are trivial $\mathbb{Z}$-modules, respectively, for all $n \geq 3(k-1)+1$ and $n \geq 2(k-1)+1$.

We have to prove that $H_k(C(\mathbf{A_n}), R_\tau)$ and $H_k(C(\mathbf{B_n}), R_\tau)$ are trivial $\mathbb{Z}$-modules, respectively, for all $n \geq 3k+1$ and $n \geq 2k+1$.

First we consider the case $n = 3k+1$ ($n = 2k+1$); using the sequence (14) ((15)), one needs only to prove that $H_k(C(\mathbf{A_{3k+1}})/\oplus_{j=1}^{m_{3k-1}} C(\mathbf{A_{3k-1}}), R_\tau)$ ($H_k(C(\mathbf{B_{2k+1}})/\oplus_{j=1}^{m_{2k}} C(\mathbf{B_{2k}}), R_\tau)$) is trivial.

The assertion in case $\mathbf{B_{2k+1}}$ follows from Lemma 4.1 since

$$H_*(C(\mathbf{B_{2k+1}})/\oplus_{j=1}^{m_{2k}} C(\mathbf{B_{2k}}), R_\tau) = H_*(F^1_{2k+1}(\mathbf{B_{2k+1}}), R_\tau)$$

and $H_{k-h}(C(\mathbf{B_{2k-h}}), R_\tau)$ is trivial for all $1 \leq h \leq k$ by inductive hypothesis.

The proof in case $\mathbf{A_{3k+1}}$ is a consequence of remark 4.1.

One has that $H_{k-1}(C(\mathbf{A_{3k-1}}), R_\tau)$ is a trivial $\mathbb{Z}$-module by induction and, from Lemma 4.1, $H_k(F^1_{3k}(\mathbf{A_{3k}}), R_\tau)$ and $H_k(F^2_{3k+1}(\mathbf{A_{3k+1}}), R_\tau)$ are trivial since $H_{k-h}(C(\mathbf{A_{3k-h-1}}), R_\tau)$ and $H_{k-h}(C(\mathbf{A_{3k-h}}), R_\tau)$ are trivial by hypothesis, respectively, for $1 \leq h \leq k$ and $2 \leq h \leq k$.

Let now $n > 3k+1$, we conclude the proof for $\mathbf{A_n}$ using induction on n.

One supposes that $H_k(C(\mathbf{A_{n-1}}), R_\tau)$ is trivial as $\mathbb{Z}$-module. Moreover $H_{k-h}(C(\mathbf{A_{n-h-1}}), R_\tau)$ are trivial by inductive hypothesis on the degree of homology, since $(n-h-1) \geq 3(k-h)+1$ for all $1 \leq h \leq k$. Then $H_{k-h}(C(\mathbf{A_{n-h-1}}), R_\tau)$ are trivial for $0 \leq h \leq k$ and the thesis follows from Lemma 4.1.

The proof in case $\mathbf{B_n}$, for $n > 2k+1$, is exactly the same.

Case $\mathbf{D_n}$ is a consequence of Lemma 4.1 applied to the exact sequence of complexes

$$0 \longrightarrow \bigoplus_{j=1}^{m_{n-1}} C(\mathbf{D_{S_{n-1}}}) \longrightarrow C(\mathbf{D_n}) \longrightarrow F^1_n(\mathbf{D_n}) \longrightarrow 0$$

since $C(\mathbf{D_{S_k}}) = C(\mathbf{A_k})$ for all $0 \leq k \leq n-1$ (we use the standard Dynking diagram of $\mathbf{D_n}$). $\square$

The last step is the

Proof of theorem 1.2 From the universal coefficients theorem it follows

$$H_k(C(\mathbf{W}),\{\varphi_1\}) \simeq H_k(C(\mathbf{W}),R_\tau)\otimes\{\varphi_1\}\oplus Tor(H_{k-1}(C(\mathbf{W}),R_\tau),\{\varphi_1\}). \tag{17}$$

If we set

$$rk_{\mathbb{Q}}(H_k(C(\mathbf{W}),R_\tau)\otimes\{\varphi_1\}) =: a_{k+1}$$

then, in the range specified in theorem 1.1

$$rk_{\mathbb{Q}}[Tor(H_{k-1}(C(\mathbf{W}),R_\tau),\{\varphi_1\})] =: a_k.$$

We recall, also, that $\{\varphi_1\}=\mathbb{Q}$, then

$$H_k(C(\mathbf{W}),\{\varphi_1\}) = H_k(C(\mathbf{W}),\mathbb{Q}),$$

moreover the rank of $H_k(C(\mathbf{W}),\mathbb{Q})$ equals the rank of $H^k(C(\mathbf{W}),\mathbb{Z})$.

It follows that relation (17) gives

$$rk[H^k(C(\mathbf{W}),\mathbb{Z})] = a_{k+1}+a_k$$

and from a simple induction

$$a_{k+1} = \sum_{i=0}^{k}(-1)^{(k-i)} rkH^i(C(\mathbf{W}),\mathbb{Z}). \quad \square$$

Remark 5.1. *With the same technique used to prove theorem 1.1, it is possible to prove a more general result.*

Let $(\mathbf{W},S)$ *be a finite Coxeter system with* $|\ S\ |= n$ *and* $m\in\mathbb{N}$ *s.t.* $m \mid o(\mathcal{A}(\mathbf{W}))$*. If there exists an integer* h *s.t.* $m \nmid o(\mathcal{A}(\mathbf{W}_k))$ *for all* $h<k<n$*, then there exists an integer* p *s.t., for all* $r<p$*,* $H^r(C(\mathbf{W}_h),R_\tau)$ *is annihilated by a squarefree element* $(1-\tau^s)$ *with* $s\mid o(\mathcal{A}(\mathbf{W}))$*,* $s<m$*, and, for all* $q<p+(n-\ h\ -\ 1)$*,* $H^q(C(\mathbf{W}),R_\tau)$ *is annihilated by a squarefree element* $(1-\tau^a)$ *with* $a\mid o(\mathcal{A}(\mathbf{W}))$*,* $a<m$*.*

As corollaries we obtain:

- $H^{q+1}(C(\mathbf{A_{3q}}), R_\tau)$ *and* $H^{q+1}(C(\mathbf{A_{3q-1}}), R_\tau)$ *are annihilated by the square-free element* $(1-\tau^3)$.

- *if* $m \mid o(\mathcal{A}(\mathbf{W}))$ *and* $m \nmid o(\mathcal{A}(\mathbf{W}_k))$ *for all* $k < n$ *then, for* $h < n$, $H^h(C(\mathbf{W}), R_\tau)$ *is annihilated by a squarefree element* $(1-\tau^s)$ *with* $s \mid o(\mathcal{A}(\mathbf{W}))$, $s < m$.

Acknowledgements *Special thanks to prof.Mario Salvetti for very useful conversations.*